SOME SOLUTIONS OF THE PELLIAN EQUATIONS $x^2 - Ay^2 = \pm 4$.

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The Pellian equations $x^2 - Ay^2 = \pm 4$, as well as the equations $x^2 - Ay^2 = \pm 1$, are of great importance and interest in the theory of numbers, and in particular in determining the units of a real quadratic domain.

The units of a quadratic domain are those integers of the domain which divide every integer of the domain. For an imaginary quadratic domain the number of integers is limited. The domain of the square root of negative one, $k(i)$, has four units, ± 1, $\pm i$; $k(\sqrt{-3})$ has six units, ± 1, $\pm (1 \pm \sqrt{-3})/2$, and all others have only two units, ± 1. But a real quadratic domain has an infinite number of units. It becomes convenient to distinguish a fundamental unit; it is the smallest unit of the domain > 1.

Now the solutions of the Pellian equations $x^2 - Ay^2 = \pm 1$ or ± 4 determine units for the domain $k(\sqrt{A})$ but the fundamental solution of the equation $x^2 - Ay^2 = 1$ does not determine the fundamental unit when the solution of the equations $x^2 - Ay^2 = -1$ or 4 or -4 is possible. While the solution of the equation $x^2 - Ay^2 = 1$ is always possible for every non-square positive integral value of A, the solution of the three other equations is not always possible for every such value of A. A necessary condition for the solution of the equations $x^2 - Ay^2 = \pm 4$ with x, y, not both even is that $A \equiv 5 \pmod 8$.

To illustrate, to obtain a unit of the domain $k(\sqrt{69})$ we might solve the Pell equation $x^2 - 69y^2 = 1$, obtaining for the smallest values, greater than 0, for x, y, $x = 7{,}775$, $y = 936$, and hence for one unit $7{,}775 + 936\sqrt{69}$. But to obtain the fundamental unit solve the equation $x^2 - 69y^2 = +4$, since this equation has a solution, and for the smallest values of x, y, get $x = 25$, $y = 3$; and the fundamental unit for the domain $k(\sqrt{69})$ is $(25 + 3\sqrt{69})/2$.

The fundamental solutions* for the Pell equation have been published up to $A = 1{,}700$; and of the equation $x^2 - Ay^2 = 4$ or the equation $x^2 - Ay^2 = -4$ up to $A = 997$.

* For account of the solutions of the Pell equations $x^2 - Ay^2 = \pm 1$, see E. E. Whitford, "The Pell Equation," New York, 1912; and for solutions of the equations $x^2 - Ay^2 = \pm 4$, see A. Cayley, "Note sur l'équation $x^2 - Dy^2 = \pm 4$, $D \equiv 5 \pmod 8$)," Journal für die reine und angewandte Mathematik, vol. 53 (1857), p. 369.

The following table gives the fundamental solutions of the equations $x^2 - Ay^2 = \pm 4$ for $A \equiv 5 \pmod 8$ from $A = 1{,}005$ to $1{,}997$, where such solutions are possible. Where the solution of $x^2 - Ay^2 = -4$ is possible, that solution is given first, followed by the solution of the equation $x^2 - Ay^2 = 4$.

Table of the Fundamental Solutions of the Equations $\mathbf{x^2 - Ay^2 = \pm 4}$, $\mathbf{A \equiv 5 \pmod 8}$ where such Solutions are Possible, from A = 1005 to A = 1997.

A	x	y
1,005	1,807	57
1,013	923	29
	851,931	26,767
1,021	85 745,895	2 683,493
	7,352 358,509 351,027	230 098,509 011,235
1,029	57,965	1,807
1,037	161	5
	25,923	805
1,045	97	3
1,061	264,395	8,117
	69,904 716,027	2,146 094,215
1,069	106 822,461	3 267,185
	11,411 038,174 096,521	348 991,093 242,285
1,077	361	11
1,085	33	1
1,093	33	1
	1,091	33
1,101	365	11
1,109	106,865	3,209
	11,420 128,227	342 929,785
1,117	7 484,589	223,945
	56 019,072 498,923	1 676,136 283,605
1,125	15,127	451
1,133	101	3
1,141	1,275 183,065	37 751,109
1,165	1,809	53
	3 272,483	95,877
1,181	29,039	845
	843 263,523	24 537,955
1,189	25,689	745
	659 924,723	19 138,305
1,197	173	5
1,205	243	7
1,221	35	1
1,229	35	1
	1,227	35
1,237	1 294,047	36,793
	1 674,557 638,211	47,611 871,271
1,245	247	7
1,253	177	5
1,261	79,011	2,225

	6,242 738,123	175 799,475
1,277	6,611	185
	43 705,323	1 223,035
1,285	25,989	725
	675 428,123	18 842,025
1,309	117,115	3,237
1,317	421,877	11,625
1,333	87,077	2,385
1,341	13 860,727	378,505
1,349	15,977	435
1,357	892,609	24,231
1,365	37	1
1,373	37	1
	1,371	37
1,381	75,401 981,961	2,029 018,105
	5,685 458,883 646,969 405,523	152 991,986 551,752 403,905
1,397	3,177	85
1,413	45,371	1,207
1,429	189	5
	35,723	945
1,437	57,961	1,529
1,453	3,059 939,997	80 274,961
	9 363,232 785,240 360,011	245,636 563,921 515,117
1,461	9 409,325	246,169
1,469	115	3
1,477	27 193,889	707,589
1,493	2,357	61
	5 555,451	143,777
1,501	3,002 570,777	77 500,215
1,509	505	13
1,517	39	1
1,525	39	1
	1,523	39
1,533	509	13
1,541	1 185,165	30,191
1,549	676,923 333,555	17,199 418,961
	458,225 199,511 213,788 938,027	11,642 688,018 289,194 536,355
1,557	29,239	741
1,565	989	25
	978,123	24,725
1,573	119	3
1,581	835	21
1,589	6 330,805	158,817
1,597	100 646,511	2 518,525
	10,129 720,176 473,123	253 480,754 116,275
1,621	4 823,622 127,875	119,806 883,557
	23 267,330 432,525 342,852 015,627	577,903 134,597 288,688 851,375
1,629	1 703,027	42,195
1,637	5,543	137
	30 724,851	759,391
1,645	26,647	657
1,653	1,423	35
1,661	2 917,473	71,585
1,669	1,293 350,265	31 658,329

	1 672,754 907,975 570,227	40,945 308,201 607,185
1,677	41	1
1,685	41	1
	1,683	41
1,693	1 372,839	33,365
	1 884,686 919,923	45,804 773,235
1,709	5 391,115	130,409
	29 064,120 943,227	703,049 916,035
1,725	623	15
1,733	172,387	4,141
	29,717 277,771	713 854,567
1,741	85,889 757,675	2,058 457,913
	7,377 050,473 470,221 405,627	176 800,451 331,756 232,275
1,749	3,973	95
1,781	211	5
	44,523	1,055
1,789	548,890 789,515	12,977 193,281
	301,281 098,814 400,033 935,227	7,123 061,865 696,843 248,715
1,797	316,873	7,475
1,821	3,821 114,165	89 543,701
1,829	81 456,873	1 904,675
1,837	28 761,577	671,055
1,845	43	1
1,853	43	1
	1,851	43
1,869	25,723	595
1,877	1,603	37
	2 569,611	59,311
1,933	812 454,627	18 479,201
	660,082 520,933 709,131	15,013 512,355 713,027
1,941	10,523 512,585	238 862,491
1,957	929	21
1,965	133	3
1,981	9 856,153 532,405	221,444 665,221
1,989	223	5
1,997	9,161	205
	83 923,923	1 878,005

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